IMSC 2048 Midterm

May 13, 2026

Excercise 1. *Below is a partial character table for a group G of order 12, with 4 conjugacy classes. The numbers in the top row are the sizes of the conjugacy classes.*

	(1) e	(3) a	(4) b	(4) c
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1			
χ_4				

where $\omega = e^{2\pi\sqrt{-1}/3}$.

(a) *Complete the character table by determining the missing entries. Please write down the orthogonality relations or formulas you use to determine the missing entries.*

(b) *Determine all normal subgroups of G from the character table. (Write down each normal subgroup as the union of certain conjugacy classes.)*

Excercise 2. *Let* $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$.

(a) *Compute $A^T A$ and find its eigenvalues and eigenvectors.*

(b) *Find the singular values $\sigma_1 \geq \sigma_2 > 0$ and write the SVD decomposition of $A = QDP^T$. (You only need to write the first two columns of Q, D, and P.)*

Excercise 3. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite dimensional representation of a finite group G over $\mathbb{C}$. If H is a subgroup of G, then the restriction $\rho|_H$ is a representation of H. Assume that ρ is the regular representation of G. Show that the restriction $\rho|_H$ is isomorphic to a direct sum of $[G:H]$ copies of the regular representation of H.*

Excercise 4. *Let W be the real vector space of Hermitian 2×2 matrices*

$$W = \{A \in M_2(\mathbb{C}) \mid A^* = A\}, \qquad A^* = \overline{A^T}.$$

(a) Prove that the rule $P \cdot A = PAP^*$ *defines a linear action of* $\mathrm{SL}_2(\mathbb{C})$ *on* W*.*

(b) Prove that the function $\langle A, A' \rangle = \det(A + A') - \det(A) - \det(A')$ *is a symmetric bilinear form on* W*, and that its signature is* $(3, 1)$*.*

(c) Use (a) and (b) to define a group homomorphism $\mathrm{SL}_2(\mathbb{C}) \to O_{3,1}$ *whose kernel is* $\{\pm I\}$*.*

Excercise 5. *Let* V *be the real vector space of Hermitian matrices in* $M_n(\mathbb{C})$

$$V = \{A \in M_n(\mathbb{C}) \mid A = \overline{A}^T\}$$

1. *Prove that* $A - \sqrt{-1}I$ *is invertible for every* $A \in V$*.*
2. *Define the map* $f\colon V \to \mathrm{GL}_n(\mathbb{C})$ *by* $f(A) = (A + \sqrt{-1}I)(A - \sqrt{-1}I)^{-1}$*. Show that* $f(A)$ *is unitary for every* $A \in V$*.*
3. *Give a characterization of the image of* $f\colon V \to \mathrm{U}(n)$ *in terms of eigenvalues, that is, which matrices can be written in the form* $(A+\sqrt{-1}I)(A-\sqrt{-1}I)^{-1}$ *for some* $A \in V$*.*
4. *Is the map* f *injective? Justify your answer.*